

# Functional Equations of Form Factors for Diagonal Scattering Theories

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## Abstract

Form factor bootstrap approach is applied for diagonal scattering theories. We consider the ADE theories and determine the functional equations satisfied by the minimal two-particle form factors. We also determine the parameterization of the singularities in two particle form factors.

For  $A_2^{(1)}$  Affine Toda field theory which is the simplest non-self conjugate theory, form factors are derived up to four-body and identification of operator is done. Generalizing this identification to the  $A_N^{(1)}$  Affine Toda cases, we fix the two particle form factors. We also determine the additional pole structure of form factors which comes from the double pole of the  $S$ -matrices of the  $A_N^{(1)}$  theory.

For  $A_N$  theories, existence of the conserved  $\mathbf{Z}_{N+1}$  charge leads to the division of the set of form factors into  $N + 1$  decoupled sectors.

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# 1 Introduction

For two dimensional factorizable scattering theories, the bootstrap framework gives strong constraints on physical quantities. For example, under some assumptions such as unitarity, the bootstrap determines the  $S$ -matrices almost completely and non-perturbatively.

The factorizable and diagonal scattering theories are integrable models which have deep relationship with underlying Lie algebras. The  $S$ -matrices for these theories are well known and determined from the data of the associated Lie algebras.

Less-known objects for these theories are form factors i.e. matrix elements of operators. If we know all form factors then the correlation function can be calculated in principle.

In integrable theories, the form factors can also be determined by the form factor bootstrap approach [1, 2].

The form factors for the diagonal scattering theories are known for the thermal perturbation of the Ising model [3, 4], the scaling Lee-Yang model [5], the sinh-Gordon model [6, 7], the Bullough-Dodd model ( $A_2^{(2)}$ ) [8] and the  $\phi_{1,3}$ -perturbed non-unitary  $M_{3,5}$  model [9] all of which contain only one type of particle, and for minimal  $A_{2N}^{(2)}$  theories which contain  $N$  kinds of particles [10]. Two-body form factors for the magnetic perturbation of the Ising model can be found in [11].

Koubek shows that for minimal  $A_{2N}^{(2)}$  theories, ( $\phi_{1,3}$ -perturbed  $M_{2,2N+3}$  minimal conformal field theories), all recursion relations for form factors can be simplified to the recursion relations for the form factors which contain only one kind of particles [12].

All particles (or excitations) in these theories are self-conjugate, i.e. a particle and its anti-particle are identical.

But for the case of non self-conjugate theories, the kinematical residue equations can not be used directly to determine the form factor. In this case, recursion relations for one particle become rather difficult to solve. In this paper, we take a straightforward approach: trying to solve simultaneously the system of recursion relations.

We mainly deal with the  $A_N^{(1)}$  models, especially the  $A_2^{(1)}$  model.

The paper is organized as follows. In section 2.1 we briefly review the form factor bootstrap approach to fix the notation. In section 2.2, we discuss the properties of minimal two particle form factors and determine the parameterization of two particle form factors. In section 3 we derive the form factors of  $A_2^{(1)}$  Affine Toda field theory up to four-body. And we identify the fundamental operators. The generalization to  $A_N$  theories are discussed in section 4, and we determine the two-particle form factors. Section 5 are conclusions and discussion.

## 2 The form factor bootstrap

### 2.1 Equations for the form factors

The matrix elements of a local operator  $\mathcal{O}(x)$

$$\begin{aligned} & F_{a_1 a_2 \dots a_n}^{a'_1 a'_2 \dots a'_m}(\beta'_1, \beta'_2, \dots, \beta'_m | \beta_1, \beta_2, \dots, \beta_n) \\ &= {}^{a'_1 a'_2 \dots a'_m} < \beta'_1, \beta'_2, \dots, \beta'_m | \mathcal{O}(0) | \beta_1, \beta_2, \dots, \beta_n >_{a_1 a_2 \dots a_n}, \end{aligned} \quad (2.1)$$

are called form factors of general kind. Here  $\beta_i$  is a rapidity of a particle of a species  $a_i$ . Consider the following form of matrix elements

$$F_{a_1 a_2 \dots a_n}(\beta_1, \beta_2, \dots, \beta_n) = \langle 0 | \mathcal{O}(0) | \beta_1, \beta_2, \dots, \beta_n \rangle_{a_1 a_2 \dots a_n} . \quad (2.2)$$

The general form factors (2.1) are related to the functions (2.2) by analytic continuation

$$\begin{aligned} & F_{a_1 \dots a_n}^{a'_1 \dots a'_m}(\beta'_1, \dots, \beta'_m | \beta_1, \dots, \beta_n) \\ &= \mathcal{C}^{a'_1 b_1} \dots \mathcal{C}^{a'_m b_m} F_{b_1 \dots b_m a_1 \dots a_n}(\beta'_1 + i\pi, \dots, \beta'_m + i\pi, \beta_1, \dots, \beta_n), \end{aligned} \quad (2.3)$$

provided that the set of rapidities  $\beta'$  are separated from the set  $\beta$ . Here  $\mathcal{C}^{ab}$  is inverse of charge conjugation matrix  $\mathcal{C}_{ab}$  and for ADE scattering theories  $\mathcal{C}_{ab} = \delta_{a\bar{b}}$ .

Watson's equations for diagonal scattering theories take a simple form

$$\begin{aligned} & F_{a_1 \dots a_i a_{i+1} \dots a_n}(\beta_1, \dots, \beta_i, \beta_{i+1}, \dots, \beta_n) \\ &= S_{a_i a_{i+1}}(\beta_i - \beta_{i+1}) F_{a_1 \dots a_{i+1} a_i \dots a_n}(\beta_1, \dots, \beta_{i+1}, \beta_i, \dots, \beta_n), \\ & F_{a_1 a_2 \dots a_n}(\beta_1 + 2\pi i, \beta_2, \dots, \beta_n) = F_{a_2 \dots a_n a_1}(\beta_2, \dots, \beta_n, \beta_1). \end{aligned} \quad (2.4)$$

The simple pole structure of the form factors are summarized by the following two types of the recursion relations.<sup>1</sup>

The first kind of relations are called kinematical residue equation

$$-i \operatorname{res}_{\beta'=\beta+i\pi} F_{a a d_1 \dots d_n}(\beta', \beta, \beta_1, \dots, \beta_n) = F_{d_1 \dots d_n}(\beta_1, \dots, \beta_n) \left( 1 - \prod_{j=1}^n S_{a d_j}(\beta - \beta_j) \right). \quad (2.5)$$

And the second kind of relations are called bound state residue equation

$$-i \operatorname{res}_{\beta'=\beta+i\theta_{ab}^c} F_{a b d_1 \dots d_n}(\beta', \beta, \beta_1, \dots, \beta_n) = \Gamma_{ab}^c F_{\bar{c} d_1 \dots d_n}(\beta + i\bar{\theta}_{bc}^a, \beta_1, \dots, \beta_n), \quad (2.6)$$

where  $\bar{\theta} = \pi - \theta$ . The on-shell three point vertex  $\Gamma_{ab}^c$  is given by

$$-i \operatorname{res}_{\beta=i\theta_{ab}^c} S_{ab}(\beta) = (\Gamma_{ab}^c)^2.$$

## 2.2 Minimal form factors

In the case of  $n = 2$ , Watson's equations reduce to

$$\begin{aligned} & F_{ab}(\beta) = S_{ab}(\beta) F_{ba}(-\beta), \\ & F_{ab}(\beta + 2\pi i) = F_{ba}(-\beta). \end{aligned} \quad (2.7)$$

The general solution of Watson's equations takes the form [1]

$$F_{a_1 \dots a_n}(\beta_1, \dots, \beta_n) = K_{a_1 \dots a_n}(\beta_1, \dots, \beta_n) \prod_{i < j} F_{a_i a_j}^{(min)}(\beta_i - \beta_j), \quad (2.8)$$

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<sup>1</sup> If  $S$ -matrices contain double poles then additional simple pole structure appears.

where  $F_{ab}^{(min)}$  is solution of eq.(2.7) which is analytic in the strip  $0 \leq Im\beta \leq 2\pi$  and has no zeros in  $0 < Im\beta < 2\pi$ .

The building block of diagonal S-matrices is  $(x)_\beta = (x)_{+(\beta)} / (-x)_{+(\beta)}$  where  $(x)_{+(\beta)} = \frac{1}{i\pi} \sinh \frac{1}{2}(\beta + i\frac{\pi}{h}x)$ . We write the basic building block of minimal (two particle) form factor corresponding to  $(x)_\beta$  as following forms

$$f_x(\beta) = \sinh \frac{1}{2}\beta \frac{g_x(\beta)}{g_{2h-x}(\beta)} \quad (2.9)$$

which has no poles and zeros in the strip  $0 < Im\beta < 2\pi$  for  $0 \leq x < 2h$ . The function  $g_x(\beta)$  is given by

$$g_x(\beta) \equiv \prod_{n=1}^{\infty} \frac{\Gamma(n + \frac{i\beta}{2\pi} - \frac{x}{2h} + 1)}{\Gamma(n - \frac{i\beta}{2\pi} + \frac{x}{2h} - 1)}. \quad (2.10)$$

The function  $g_x(\beta)$  has poles at  $\beta = -i\pi x/h + 2(m+2)\pi i$  for  $m = 0, 1, 2, \dots$  and zeros at  $\beta = -i\pi x/h - 2m\pi i$  for  $m = 0, 1, 2, \dots$

The introduction of the function  $g_x$  simplifies the calculation. Using the following properties of  $g_x$

$$g_x\left(\beta + i\frac{\pi}{h}y\right) = g_{x+y}(\beta), \quad (2.11)$$

$$g_x(\beta + 2\pi i) = g_{x+2h}(\beta) = \frac{1}{(x)_+} g_x(\beta), \quad (2.12)$$

$$g_x(i\pi - \beta) = \frac{1}{g_{3h-x}(\beta)} = \frac{(h-x)_+}{g_{h-x}(\beta)}, \quad (2.13)$$

$$g_x(0)g_{4h-x}(0) = 1, \quad (2.14)$$

the behavior of the minimal form factors under the recursion relation is easily determined.

For ADE scattering theories, the basic building blocks of the diagonal S-matrices are  $< x >_\beta = < x >_{+(\beta)} / < -x >_{+(\beta)}$  with

$$< x >_+ = \begin{cases} (x-1)_+(x+1)_+ & \text{for perturbed conformal} \\ \frac{(x-1)_+(x+1)_+}{(x-1+B)_+(x+1-B)_+} & \text{for Affine Toda} \end{cases}$$

where  $h$  is the Coxeter number of the associated Lie algebra.

So the building block corresponding to  $< x >$  is

$$F_x^{(min)}(\beta) = \frac{G_x(\beta)}{G_{2h-x}(\beta)}, \quad (2.15)$$

where

$$G_x(\beta) = \begin{cases} g_{x-1}(\beta)g_{x+1}(\beta) & \text{for perturbed conformal} \\ \frac{g_{x-1}(\beta)g_{x+1}(\beta)}{g_{x-1+B}(\beta)g_{x+1-B}(\beta)} & \text{for Affine Toda.} \end{cases}$$

We list some properties of  $G_x$ .

$$G_x\left(\beta + i\frac{\pi}{h}y\right) = G_{x+y}(\beta), \quad (2.16)$$

$$G_x(\beta + 2\pi i) = G_{x+2h}(\beta) = \frac{1}{< x >_+} G_x(\beta), \quad (2.17)$$

$$G_x(i\pi - \beta) = \frac{1}{G_{3h-x}(\beta)} = \frac{< h - x >_+}{G_{h-x}(\beta)}. \quad (2.18)$$

$$G_x(0)G_{4h-x}(0) = 1. \quad (2.19)$$

If  $S$ -matrices have the following form

$$S_{ab}(\beta) = \prod_{x \in A_{ab}} < x >_\beta, \quad (2.20)$$

then the minimal solutions of eq.(2.7) are written as

$$F_{ab}^{(min)}(\beta) = \prod_{x \in A_{ab}} F_x^{(min)}(\beta). \quad (2.21)$$

Here  $A_{ab}$  is the minimal set of numbers which gives the correct  $S_{ab}$ . The multiplicity of  $p$  in  $A_{ab}$  is denoted by  $m_p(A_{ab})$ . The sets  $A_{ab}$  are chosen such that  $m_p(A_{ab}) = 0$  for  $p \leq 0$  or for  $p \geq h$ .

The crossing condition of the  $S$ -matrix  $S_{ab}(i\pi - \beta) = S_{b\bar{a}}(\beta)$  is equivalent to the condition  $m_{h-p}(A_{ab}) = m_p(A_{b\bar{a}})$ .

The minimal form factor (2.21) has following property

$$F_{\bar{a}b}^{(min)}(\beta + i\pi)F_{ab}^{(min)}(\beta) = 1/\xi_{ab}(\beta). \quad (2.22)$$

Eq.(2.22) is an analogue of the  $S$ -matrix relation  $S_{\bar{a}b}(\beta + i\pi)S_{ab}(\beta) = 1$ . There is one-to-one correspondence between the constituent of minimal form factors and the  $S$ -matrices. Many properties of  $F_x^{(min)}$  are similar to those of  $< x >$  but the monodromy property is quite different.  $F_x^{(min)}$  has a diagonal monodromy which comes from eq.(2.17) while  $< x >$  is  $2\pi i$ -periodic. Due to these monodromy factors, additional functions  $\xi_{ab}(\beta)$  appear in the right hand side of eq.(2.22). The factor  $1/\xi_{ab}$  is given by the product of  $< x >_+$

$$1/\xi_{ab}(\beta) = \prod_{x \in A_{ab}} < x >_{+(\beta)}. \quad (2.23)$$

In particular, for Affine cases,  $< x >_+ = (x-1)_+(x+1)_+/(x-1+B)_+(x+1-B)_+$  implies that  $B$  dependent parts appear only in the numerator of  $\xi_{ab}$ . The form of  $\xi$  has been determined for the scaling Lee-Yang model [5], the sinh-Gordon model [7] and for the Bullough-Dodd model [8] by explicit calculation.

Using eq. (2.7), we get the following relations:

$$S_{ab}(\beta) = \frac{\xi_{\bar{a}b}(\beta + i\pi)}{\xi_{ab}(\beta)} = \frac{\xi_{ab}(-\beta)}{\xi_{ab}(\beta)}. \quad (2.24)$$

Under the factorization of eq.(2.8), the kinematical residue equations reduce to

$$\begin{aligned} & -i \operatorname{res}_{\beta'=\beta+i\pi} K_{\bar{a}ad_1\dots d_n}(\beta', \beta, \beta_1, \dots, \beta_n) \\ &= K_{d_1\dots d_n}(\beta_1, \dots, \beta_n) \left( \prod_{j=1}^n \xi_{ad_j}(\beta - \beta_j) - \prod_{j=1}^n \xi_{ad_j}(\beta_j - \beta) \right) / F_{\bar{a}a}^{(min)}(i\pi). \end{aligned} \quad (2.25)$$

In accordance with the  $S$ -matrix bootstrap

$$S_{ad}(\beta + i\bar{\theta}_{ac}^b) S_{bd}(\beta - i\bar{\theta}_{bc}^a) = S_{\bar{c}d}(\beta), \quad (2.26)$$

the minimal form factors have following properties

$$F_{ad}^{(min)}(\beta + i\bar{\theta}_{ac}^b) F_{bd}^{(min)}(\beta - i\bar{\theta}_{bc}^a) = F_{\bar{c}d}^{(min)}(\beta) / \lambda_{ab;d}^c(\beta). \quad (2.27)$$

The extra factor  $1/\lambda_{ab;d}^c(\beta)$  comes from the diagonal monodromy and is given by the product of  $\langle x \rangle_+$ :

$$1/\lambda_{ab;d}^c(\beta) = \prod_{\{x \in A_{ad} | x < \bar{u}_{ac}^b\}} \langle \bar{u}_{ac}^b - x \rangle_{+(\beta)} \prod_{\{x \in A_{bd} | x < \bar{u}_{bc}^a\}} \langle x - \bar{u}_{bc}^a \rangle_{+(\beta)}. \quad (2.28)$$

Here  $u_{ab}^c = h\theta_{ab}^c/\pi$  and  $\bar{u}_{ab}^c = h - u_{ab}^c$ .

We list some properties of  $\lambda_{ab;d}^c$ :

$$\lambda_{ab;d}^c(\beta) = \lambda_{ba;d}^c(-\beta),$$

$$\lambda_{ca;\bar{d}}^b(\beta - i\theta_{ab}^c) \lambda_{ab;d}^c(\beta - i\bar{\theta}_{ac}^b) = \xi_{ad}(-\beta).$$

The function  $\lambda$  is known for the scaling Lee-Yang model [5] and the Bullough-Dodd model [8, 13].

The Bound state residue equations reduce to

$$\begin{aligned} & -i \operatorname{res}_{\beta'=\beta+i\theta_{ab}^c} K_{abd_1\dots d_n}(\beta', \beta, \beta_1, \dots, \beta_n) \\ &= \Gamma_{ab}^c K_{\bar{c}d_1\dots d_n}(\beta + i\bar{\theta}_{bc}^a, \beta_1, \dots, \beta_n) \prod_{j=1}^n \lambda_{ab;d_j}^c(\beta + i\bar{\theta}_{bc}^a - \beta_j) / F_{ab}^{(min)}(i\theta_{ab}^c). \end{aligned} \quad (2.29)$$

For ADE scattering theories, the  $S$ -matrix can be written as [14]

$$S_{ab}(\beta) = \prod_{p=0}^{h-1} \left( \langle 2p+1 + \epsilon_{ab} \rangle_{+(\beta)} \right)^{\mu^{(a)} \cdot w^{-p} \phi_b}. \quad (2.30)$$

Or equivalently

$$m_x(A_{ab}) = \mu^{(a)} \cdot w^{-p} \phi_b \quad \text{for } 0 < x = 2p+1 + \epsilon_{ab} < h.$$

Here  $\mu^{(a)}$  is the fundamental weight of the algebra. So the minimal form factors can be given by<sup>2</sup>

$$F_{ab}^{(min)}(\beta) = \langle 0 \rangle_+^{\delta(\epsilon_{ab}, 1) \mu^{(a)} \cdot w \phi_b} \prod_{p=0}^{h-1} (G_{2p+1+\epsilon_{ab}}(\beta))^{\mu^{(a)} \cdot w^{-p} \phi_b}. \quad (2.31)$$

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<sup>2</sup> For the minimal cases, if the  $S$ -matrix have fermionic nature:  $S_{ab}(0) = -1$ , then one more factor  $(0)_{+(\beta)}$  is needed.

Here  $\epsilon_{ab} = \frac{1}{2}(c(a) - c(b))$ . Depending the two colourings of the Dynkin diagram of the algebra associated,  $c(a) = 1$  for white nodes and  $c(a) = -1$  for black nodes [15].

Note that we can see  $c(\bar{a}) = (-1)^h c(a)$ . If the Coxeter number  $h$  is even, which holds except for  $A_{2N}$  theories,  $\epsilon_{\bar{a}b} = \epsilon_{ab}$ .

The first adjustment factor in the right hand side of eq. (2.31) is introduced in order that  $F_{ab}^{(min)}$  is constructed from  $G_x$  ( $0 \leq x < 2h$ ) for  $\epsilon_{ab} = 1$ .

In the right hand side of eq.(2.25) and eq.(2.29), for the Affine cases, coupling dependent parts  $(x \pm B)_+$  only appear in positive powers, so the singularities of  $K$  can be factorized by products of  $1/(x)_+$ .

$$K_{a_1 \dots a_n}(\beta_1, \dots, \beta_n) = Q_{a_1 \dots a_n}(\beta_1, \dots, \beta_n) \prod_{i < j} \frac{1}{(e^{\beta_i} + e^{\beta_j})^{c_{a_i a_j}} W_{a_i a_j}(\beta_i - \beta_j)}. \quad (2.32)$$

The function  $W_{ab}$  contains the factor  $(u_{ab}^c)_+(-u_{ab}^c)_+$  for bound state poles. The factor  $(-u_{ab}^c)_+$  is needed to make  $W_{ab}$  symmetric:  $W_{ba}(-\beta) = W_{ab}(\beta)$ . In general,  $W_{ab}$  must contain more factors to factorize the higher order poles in  $\xi_{ab}$ .

The polynomial  $Q_{a_1 \dots a_n}$  carries the information about operators. Counting the number of independent solution  $Q_{a_1 \dots a_n}$  can be used to classify the operator content of the model [3, 6, 12, 21, 16].

We expect that  $Q_{a_1, \dots, a_n}$  are polynomials in  $(x)_+(-x)_+$ . So the function  $W_{ab}$  is determined from the requirement that  $W_{\bar{a}b}(\beta + i\pi)W_{ab}(\beta)\xi_{ab}(\beta)$  is product of the factors  $(x)_+$  in positive powers.

Using the expression (2.23), we write the singularity of  $\xi_{ab}$  as follows

$$\prod_{x \in A_{ab}} \frac{1}{(x-1)_+(x+1)_+} = \prod_{p=0}^h (p)_+^{-m_{p-1}(A_{ab}) - m_{p+1}(A_{ab})}.$$

Even order poles do not correspond to the bound state. If  $\xi_{ab}$  contain the factor  $1/(x)_+^{2k}$ , the half  $1/(x)_+^k$  is canceled by  $W_{ab}(\beta)$  and the other half is canceled by  $W_{\bar{a}b}(\beta + i\pi)$ . Then  $W_{ab}(\beta)$  must contain  $(x)_+^k(-x)_+^k$ , and  $W_{\bar{a}b}(\beta)$  have  $(x-h)_+^k(h-x)_+^k$ .

Odd order poles can be interpreted as the production of a bound state.  $W_{ab}$  contains the factor  $(u_{ab}^c)_+(-u_{ab}^c)_+$  for the bound state in forward channel. If  $\xi_{ab}$  contains the factor  $1/(x)_+^{2k+1}$  then  $W_{ab}$  has the factor  $(x)_+^{k+1}$  and  $W_{\bar{a}b}$  has the factor  $(x-h)_+^k$  for the forward channel.

Dorey's "uphill/downhill" mnemonic [14] implies that  $m_{p-1} - m_{p+1} = +1, 0, -1$  and the case  $m_{p-1} - m_{p+1} = +1$  corresponds to the forward channel.

The above consideration leads to the following form of the parameterization:

$$W_{ab}(\beta) = \rho_{ab} \left( \beta + i\frac{\pi}{h} \right) \rho_{ab} \left( -\beta + i\frac{\pi}{h} \right),$$

where

$$\rho_{ab}(\beta) = \prod_{\{x \in A_{ab} | x \neq h-1\}} (x)_{+(\beta)}.$$

The singularity structure of  $\xi_{ab}$  is similar to that of  $S_{ab}$ . The singularities of the  $S$ -matrices for ADE theories are explained in terms of multi-scattering processes [17, 18]. So it is natural to expect that the factorization in the above admits such interpretation.

For perturbed conformal theories, Delfino and Mussardo [11] have derived the parameterization of the singularities of two particle form factors from the Feynman diagrammatic analysis of multi-particle processes. Their factorization agrees with our result.

### 3 $A_2^{(1)}$ case

The  $A_2^{(1)}$  theory is the simplest model based on the Lie algebra which contain non self-conjugate particles. It contains two kinds of particles, which are denoted by 1 and 2. The particle 2 is the anti-particle of 1 and vice versa.

The  $S$ -matrices of this theory are given by  $S_{11} = S_{22} = \langle 1 \rangle$  and  $S_{12} = S_{21} = \langle 2 \rangle$  [18, 19, 20]. For definiteness, we consider the Affine  $A_2^{(1)}$  theories. We define

$$\begin{aligned} & F_{[m,n]}(\beta_1, \beta_2, \dots, \beta_m; \beta'_1, \beta'_2, \dots, \beta'_n) \\ &= F_{11\dots 122\dots 2}(\beta_1, \beta_2, \dots, \beta_m, \beta'_1, \beta'_2, \dots, \beta'_n). \end{aligned} \quad (3.1)$$

And we factorize  $K_{[m,n]}$  as follows

$$\begin{aligned} & K_{[m,n]}(\beta_1, \dots, \beta_m; \beta'_1, \dots, \beta'_n) \\ &= \frac{Q_{[m,n]}(x_1, \dots, x_m; y_1, \dots, y_n)}{\prod_{i < j} (x_i - \omega^2 x_j)(x_i - \omega^{-2} x_j) \prod_i \prod_j (x_i + y_j) \prod_{i < j} (y_i - \omega^2 y_j)(y_i - \omega^{-2} y_j)} \end{aligned} \quad (3.2)$$

with  $x_i = e^{\beta_i}$ ,  $y_i = e^{\beta'_i}$  and  $\omega = e^{i\frac{\pi}{3}}$ . The degree of polynomial  $Q_{[m,n]}$  is given by  $\deg(Q_{[m,n]}) = (m+n)(m+n-1) - mn$ . For simplicity, we use the vector notation  $\mathbf{x} = (x_1, x_2, \dots)$  etc.

Then the kinematical residue equation is reduced to

$$Q_{[m+1,n+1]}(\mathbf{x}, -x; x, \mathbf{y}) = x D_{[m,n]}(x; \mathbf{x}, \mathbf{y}) Q_{[m,n]}(\mathbf{x}; \mathbf{y}), \quad (3.3)$$

where

$$\begin{aligned} & D_{[m,n]}(x; \mathbf{x}, \mathbf{y}) = (-1)^n H_2 \\ & \times \left( D_m(x; \mathbf{x}; -\omega) D_n(x; \mathbf{y}; \omega^{-1}) - D_m(x; \mathbf{x}; -\omega^{-1}) D_n(x; \mathbf{y}; \omega) \right). \end{aligned}$$

The function  $D_m$  is defined by

$$\begin{aligned} D_m(x; \mathbf{x}; z) &= \prod_{j=1}^m (x + z x_j)(x - z q x_j)(x - z q^{-1} x_j) \\ &= \sum_{l=0}^m \sum_{k=0}^m \sum_{r=0}^{m-k} (-1)^k [k+1]_q x^{3m-2r-k-l} z^{2r+k+l} \sigma_l(\mathbf{x}) s_{(2r, 1^k)}(\mathbf{x}), \end{aligned} \quad (3.4)$$

where  $[k]_q = (q^k - q^{-k})/(q - q^{-1})$  and the Schur function  $s_{(2r, 1^k)}$  is

$$s_{(2r, 1^k)} = (\sigma_{r+k} \sigma_{r-2} - \sigma_{r+k-1} \sigma_{r-1}).$$

Here  $\sigma_j$  are the elementary symmetric polynomials defined by

$$\prod_{j=1}^m (x + x_j) = \sum_{j=0}^m x^{m-j} \sigma_j(\mathbf{x}).$$



The  $B$  dependent parameter  $H_2$  is defined by

$$H_2 = \frac{(-i)}{F_{12}^{(min)}(i\pi)}.$$

The coupling dependent parameter  $q = e^{i\frac{\pi}{3}(B-1)}$  transforms into  $q^{-1}$  under the weak-strong transformation  $B \rightarrow 2 - B$ .

The bound state residue equations are reduced to

$$\begin{aligned} Q_{[m+2,n]}(\mathbf{x}, \omega y, \omega^{-1}y; \mathbf{y}) &= Hy^2 D_m(y; \mathbf{x}; 1) Q_{[m,n+1]}(\mathbf{x}; y, \mathbf{y}), \\ Q_{[m,n+2]}(\mathbf{x}; \omega x, \omega^{-1}x, \mathbf{y}) &= Hx^2 D_m(x; \mathbf{y}; 1) Q_{[m+1,n]}(\mathbf{x}, x; \mathbf{y}), \end{aligned} \quad (3.5)$$

where

$$H = -\frac{\sqrt{3}\Gamma}{F_{11}^{(min)}(\frac{2}{3}\pi i)} = -\frac{\sqrt{3}\Gamma}{F_{22}^{(min)}(\frac{2}{3}\pi i)}.$$

The function  $\Gamma$  is given by

$$(\Gamma)^2 = (\Gamma_{11}^1)^2 = (\Gamma_{22}^2)^2 = \sqrt{3} \frac{\sin \frac{\pi}{6} B \sin \frac{\pi}{6} (2-B)}{\sin \frac{\pi}{6} (4-B) \sin \frac{\pi}{6} (2+B)}. \quad (3.6)$$

There is the relation between  $H$  and  $H_2$

$$H^2/H_2 = (1+\omega)(1+\omega[2]_q), \quad (3.7)$$

which is equivalent to the minimal form factor relation

$$\left(F_{11}^{(min)}(\frac{2}{3}\pi i)\right)^2 / F_{12}^{(min)}(i\pi) = < 3 >_{+(0)}. \quad (3.8)$$

We first pay attention only to the indices  $[m, n]$ . The bound state residue equations relate  $Q_{[m,n]}$  to  $Q_{[m-2,n+1]}$  or  $Q_{[m+1,n-2]}$  and the kinematical ones relate  $Q_{[m,n]}$  to  $Q_{[m-1,n-1]}$ . We identify  $[m, n]$  with two dimensional vector and introduce the following equivalence relations

$$[m, n] \sim [m, n] - l_1[2, -1] - l_2[-1, 2] - l_3[1, 1] \quad (3.9)$$

where  $l_i \in \mathbf{Z}$ . Because  $[1, 1]$  are equal to  $[2, -1] + [-1, 2]$ , the above definition is redundant. But for later convenience, we write here the  $[1, 1]$  term.

Then there are three equivalence classes

$$\{[m, n]\} / \sim = [[0, 0]] + [[1, 0]] + [[0, 1]]. \quad (3.10)$$

In other words, if  $[m, n]$  and  $[m', n']$  belong to different classes then  $Q_{[m,n]}$  and  $Q_{[m',n']}$  are not linked by residue equations.

So the set of form factors is divided into three sectors. In each sector, higher polynomials are determined iteratively from lower ones. In the minimal polynomial space, the solutions can be determined uniquely except for the kernel ambiguity.

Note that  $[2, -1]$  and  $[-1, 2]$  are equal to the first and second rows of the Cartan matrix of  $A_2$  algebra respectively.

The index  $[m, n]$  can be identified with the Dynkin indices of weights and with corresponding weights  $\mu$

$$\mu = [m_1, m_2] = m_1\mu^{(1)} + m_2\mu^{(2)}, \quad (3.11)$$

where  $\mu^{(a)}$  are the fundamental weights. Then the equivalence relation can be rewritten as follows

$$\mu \sim \mu - l_1\alpha_1 - l_2\alpha_2 - l_3(\alpha_1 + \alpha_2). \quad (3.12)$$

Here  $\alpha_1$  and  $\alpha_2$  are the simple roots of the  $A_2$  algebra. For the  $A_2^{(1)}$  theory, the equivalence relation is generated by all positive roots.

In the polynomial space we are considering, the degree of a polynomial is equal to the degree of the kernel and the kernel is one dimensional. So at every recursion step, only one parameter enter to the solution space and the parameter  $A_\nu$  is attached to the point  $\nu$  in the dominant weight lattice. The most general solution of the residue equations (3.3) and (3.5) has following forms

$$Q_\mu(\mathbf{x}, \mathbf{y}) = \sum_{\nu \leq \mu} H^{|\mu - \nu|} A_\nu Q_{\mu, \nu}(\mathbf{x}, \mathbf{y}). \quad (3.13)$$

Here the ordering of weights  $\mu \geq \nu$  means that the difference  $\mu - \nu$  lies in the dominant root lattice and  $|\mu| = m_1 + m_2$  for  $\mu = [m_1, m_2]$ .

The polynomial  $Q_\mu$  carries the information about operators. Each  $Q_{\mu, \nu}$  satisfies the residue equations, so gives an independent form factor of some operator  $\mathcal{O}_\nu$ .

### 3.1 The $[0, 0]$ -sector

We start from the  $[0, 0]$ -sector to solve the recursion relations.

Although we call  $[0, 0]$ -sector, the kinematical recursion relation is not applied to the  $Q_{[1, 1]} \rightarrow Q_{[0, 0]}$ <sup>3</sup>.

So the first polynomial is  $Q_{[1, 1]}$ . The most general degree 1 polynomial is

$$Q_{[1, 1]}(x, y) = A_{[1, 1]}x + A'_{[1, 1]}y \quad (3.14)$$

where  $A_{[1, 1]}$  and  $A'_{[1, 1]}$  are constants. But the recursion relation  $Q_{[3, 0]} \rightarrow Q_{[1, 1]}$  has no solution unless  $A_{[1, 1]} = A'_{[1, 1]}$ . Similar phenomena occur at higher stages of recursion processes. These additional constraints come from  $\mathbf{Z}_2$  symmetry corresponding to the charge conjugation. These  $\mathbf{Z}_2$  constraints are imposed on the constants:  $A_{[m, n]} = A_{[n, m]}$ .

First few solutions in the  $[0, 0]$ -sector are given by

$$Q_{[1, 1]}(x, y) = A_{[1, 1]}(x + y).$$

$$Q_{[3, 0]} = A_{[3, 0]}B_{1[3, 0]} + HA_{[1, 1]}\sigma_1\sigma_2(\sigma_1\sigma_2 - (2 + [2]_q)\sigma_3).$$

$$Q_{[2, 2]} = A_{[2, 2]}B_{1[2, 2]}K_{[2, 2]}B_{2[2, 2]} + HA_{[3, 0]}Q_{[2, 2], [3, 0]} + H^2A_{[1, 1]}Q_{[2, 2], [1, 1]}.$$

---

<sup>3</sup>If the recursion process was started from  $Q_{[0, 0]}$  formally, we would only get  $Q_{[m, n]} = Q_{[0, 0]}\delta_{m, 0}\delta_{n, 0}$ . This solution corresponds to the 'form factor of identity operator'.

Here

$$\begin{aligned} Q_{[2,2],[0,3]} &= (\sigma_1^2(x)\sigma_1^2(y) - \sigma_2(x)\sigma_2(y)) \\ &\times ((\sigma_2(x) - \sigma_2(y))^2 + \sigma_1(x)\sigma_1(y)(\sigma_2(x) + \sigma_2(y)) + \sigma_2(x)\sigma_1^2(y) + \sigma_1^2(x)\sigma_2(y)), \end{aligned}$$

$$\begin{aligned} Q_{[2,2],[1,1]} &= (\sigma_1(x) + \sigma_1(y))(\sigma_2(x)\sigma_1(y) + \sigma_1(x)\sigma_2(y)) \\ &\times (\sigma_1(x)\sigma_1(y)(\sigma_2(x) + \sigma_2(y) + \sigma_1(x)\sigma_1(y)) - (1 + [2]_q)\sigma_2(x)\sigma_2(y)). \end{aligned}$$

The polynomial  $K_{[m,n]}$  is given by

$$K_{[m,n]}(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^m \prod_{j=1}^n (x_i + y_j) = \sum_{\lambda} s_{\lambda}(\mathbf{x}) s_{\hat{\lambda}}(\mathbf{y}), \quad (3.15)$$

which is the kernel of the kinematical residue equations. Here  $s_{\lambda}$  is the Schur functions and the summation is taken for the partition  $\lambda = (\lambda_1, \dots, \lambda_m)_{\lambda}$  i.e. non-increasing sequences of non-negative integers under the condition  $\lambda_1 = n$  and  $\hat{\lambda} = (m - \lambda'_n, \dots, m - \lambda'_1)[22]$ . The partition  $\lambda'$  is the conjugate of the partition  $\lambda$ .

The polynomials  $B_{a[m,n]}(a = 1, 2)$  are given by

$$B_{1[m,n]}(\mathbf{x}, \mathbf{y}) = \prod_{i < j \leq m} (x_i - \omega^2 x_j)(x_i - \omega^{-2} x_j) = s_{2\delta_m}(\mathbf{x}), \quad (3.16)$$

$$B_{2[m,n]}(\mathbf{x}, \mathbf{y}) = \prod_{i < j \leq n} (y_i - \omega^2 y_j)(y_i - \omega^{-2} y_j) = s_{2\delta_n}(\mathbf{y}), \quad (3.17)$$

which are the kernel of the bound state residue equations. Here  $\delta_m = (m-1, m-2, \dots, 1)$ .

The Schur function can be expressed as [22]

$$s_{\lambda} = \det \left( \sigma_{\lambda'_i - i + j} \right)_{1 \leq i, j \leq l(\lambda')},$$

where  $l(\lambda)$  is the length of the partition  $\lambda$ .

The polynomials  $Q_{[m,n],[1,1]}$  ( $[m,n] \neq [1,1]$ ) have the following forms

$$Q_{[m,n],[1,1]}(\mathbf{x}; \mathbf{y}) = (\sigma_1(\mathbf{x}) + \sigma_1(\mathbf{y}))(\sigma_{m-1}(\mathbf{x})\sigma_n(\mathbf{y}) + \sigma_m(\mathbf{x})\sigma_{n-1}(\mathbf{y}))P_{[m,n]}(\mathbf{x}; \mathbf{y}). \quad (3.18)$$

From the stress-energy conservation, it is possible to show that the polynomials which enter the form factors of the trace of the stress-energy tensor  $\Theta$  are factorized as (3.18). So the operator  $\mathcal{O}_{[1,1]}$  is identified with  $\Theta$ . This fixes the constant  $A_{[1,1]}$  to be

$$A_{[1,1]} = \frac{\pi M^2}{2F_{12}^{(min)}(i\pi)},$$

where  $M$  is the mass of particles.

In this sector, we determined the form factor up to four-body ones. For example, the explicit form of two-body form factor is given as

$$F_{12}(\beta) = F_{12}^{\Theta}(\beta) = \frac{\pi M^2}{2} \frac{F_{12}^{(min)}(\beta)}{F_{12}^{(min)}(i\pi)}. \quad (3.19)$$

### 3.2 The $[1, 0]$ -sector and the $[0, 1]$ -sector

The solutions in the  $[0, 1]$ -sector are simply obtained from the  $[1, 0]$ -sector using  $\mathbf{Z}_2$  symmetry. So we only deal with the  $[1, 0]$ -sector.

First few solutions in the  $[1, 0]$ -sector is given by

$$Q_{[1,0]} = A_{[1,0]}.$$

$$Q_{[0,2]} = A_{[0,2]}B_{2[0,2]} + HA_{[1,0]}\sigma_2^{(2)}.$$

$$Q_{[2,1]} = A_{[2,1]}B_{2[2,1]}K_{[2,1]} + HA_{[0,2]}\sigma_2^{(2)}K_{[2,1]} + H^2A_{[1,0]}\sigma_1^{(2)}\sigma_2^{(2)}\sigma_1^{(1)}.$$

$$\begin{aligned} Q_{[4,0]} &= A_{[4,0]}B_{1[4,0]} + HA_{[2,1]}Q_{[4,0],[2,1]} + H^2A_{[0,2]}Q_{[4,0],[0,2]} \\ &+ H^3A_{[1,0]}Q_{[4,0],[1,0]}. \end{aligned}$$

$$\begin{aligned} Q_{[1,3]} &= A_{[1,3]}B_{2[1,3]}K_{[1,3]} + HA_{[2,1]}Q_{[1,3],[2,1]} + H^2A_{[0,2]}Q_{[1,3],[0,2]} \\ &+ H^3A_{[1,0]}Q_{[1,3],[1,0]}. \end{aligned}$$

The explicit forms of the polynomials  $Q_{[m,n],[m',n']}$  in the above equations are given by

$$\begin{aligned} Q_{[4,0],[2,1]} &= (2 + [2]_q)\sigma_2(-\sigma_1\sigma_3^3 - \sigma_1^3\sigma_3\sigma_4 + 2\sigma_1\sigma_2\sigma_3\sigma_4 + \sigma_3^2\sigma_4 + \sigma_1^2\sigma_4^2) \\ &+ \{\sigma_2^2 + (2[2]_q + [2]_q^2)\sigma_4\}(\sigma_1^2\sigma_3^2 - \sigma_2^2\sigma_4 - 2\sigma_1\sigma_3\sigma_4 + \sigma_4^2). \end{aligned}$$

$$Q_{[4,0],[0,2]} = \sigma_4 \left( (2 + [2]_q)(\sigma_4 - \sigma_1\sigma_3)(\sigma_4 + \sigma_2^2 - \sigma_1\sigma_3) + ([3]_q - 5)\sigma_2^2\sigma_4 \right).$$

$$\begin{aligned} Q_{[4,0],[1,0]} &= \sigma_4 \left( (2 + [2]_q)\sigma_2(-\sigma_3^2 - \sigma_1^2\sigma_4 + \sigma_2\sigma_4) \right. \\ &\quad \left. + (\sigma_2^2 + ([3]_q + 3[2]_q + 1)\sigma_4)(\sigma_1\sigma_3 - \sigma_4) \right). \end{aligned}$$

$$Q_{[1,3],[1,0]} = \sigma_3^{(3)}\sigma_1^{(1)} \left( \sigma_1^{(3)}\sigma_2^{(3)} - (2 + [2]_q)\sigma_3^{(3)} \right) \left( \sigma_2^{(3)} + \sigma_1^{(3)}\sigma_1^{(1)} \right).$$

$$Q_{[1,3],[0,2]} = \sigma_1^{(1)} \left( \sigma_1^{(3)}\sigma_2^{(3)} - (2 + [2]_q)\sigma_3^{(3)} \right) \left( (\sigma_1^{(3)})^2\sigma_3^{(3)} + (\sigma_2^{(3)})^2\sigma_1^{(1)} \right).$$

$$Q_{[1,3],[2,1]} = \sigma_1^{(3)}\sigma_2^{(3)} \left( \sigma_1^{(3)}\sigma_2^{(3)} - (2 + [2]_q)\sigma_3^{(3)} \right) K_{[1,3]}.$$

The form factor of the fundamental operator is factorized as follows

$$Q_{\mu,\nu}(\mathbf{x}, \mathbf{y}) = \sigma_m^{(m)}(\mathbf{x})\sigma_n^{(n)}(\mathbf{y})P_{\mu,\nu}(\mathbf{x}, \mathbf{y}). \quad (3.20)$$

The operators  $\mathcal{O}_{[1,0]}$  and  $\mathcal{O}_{[0,1]}$  correspond to these fundamental operators, and others are composite operators.

Note that  $[1, 0]$  and  $[0, 1]$  are fundamental weights of  $A_2$  algebra. The operators labeled by the fundamental weight correspond to the fundamental operators. From the conservation of the  $\mathbf{Z}_3$  charge,  $\mathcal{O}_{\mu^{(a)}} = \phi_{\bar{a}}$  ( $a = 1, 2$ ). Here  $\phi_a$  are Affine Toda fields.

The requirement

$$F_a(\beta) = F_a^{\phi_{\bar{a}}}(\beta) = \langle 0 | \phi_{\bar{a}}(0) | \beta \rangle_a = \frac{1}{\sqrt{2}} \quad (a = 1, 2)$$

fix the constant  $A_{[1,0]} = A_{[0,1]}$  to be  $1/\sqrt{2}$ .

In  $[1, 0]$ -sector and  $[0, 1]$ -sector, we also determined the form factors up to four-body ones. For example, the two-body form factor is given by

$$F_{22}^{\phi_2}(\beta) = -\sqrt{\frac{3}{2}} \frac{\Gamma}{2 \cosh \beta + 1} \frac{F_{22}^{(min)}(\beta)}{F_{22}^{(min)}(\frac{2}{3}\pi i)}.$$

## 4 The $A_N$ case

The  $A_N$  theory contains  $N$  kinds of particles, which are denoted by  $1, \dots, N$ . The anti-particle of  $a$  is  $\bar{a} = h - a = N + 1 - a$ . The mass of the particle of type  $a$  is given by  $M_a = 2M \sin(a\pi/h)$ [19]. As for the case of the  $A_2$  theory, the  $A_N$  form factors are divided into  $N + 1$  sectors. The sector specified by the fundamental sector contains the fundamental operator, and the zero sector contains the identity operator and the stress-energy operator.

Physically, these sectors simply come from the decomposition of the states into the different  $\mathbf{Z}_{N+1}$ -charge sectors.

For the  $A_N$  theory, the explicit form of  $F_{ab}^{(min)}$  can be written as

$$F_{ab}^{(min)}(\beta) = \prod_{\substack{x=|a-b|+1 \\ \text{step } 2}}^{a+b-1} F_x^{(min)}(\beta) = \prod_{\substack{x=|a-b|+1 \\ \text{step } 2}}^{h-|a+b-h|-1} F_x^{(min)}(\beta).$$

Using  $F_x^{(min)}(\beta) F_{2h-x}^{(min)}(\beta) = 1$ , we can show  $F_{ab}^{(min)}(\beta) = F_{\bar{a}\bar{b}}^{(min)}(\beta)$ .

The monodromy factors  $\xi$  and  $\lambda$  are given as follows

$$1/\xi_{ab}(\beta) = \prod_{\substack{x=|a-b|+1 \\ \text{step } 2}}^{h-|a+b-h|-1} \langle x \rangle_{+(\beta)},$$

$$1/\lambda_{ab;d}^c(\beta) = \begin{cases} \prod_{\substack{v(a,b,d)+1 \\ \text{step } 2}}^{b-|a-d|-1} \langle x \rangle_{+(\beta)} & \text{for } a + b + c = h \\ \prod_{\substack{v(\bar{a},\bar{b},\bar{d})+1 \\ \text{step } 2}}^{\bar{b}-|\bar{a}-\bar{d}|-1} \langle x \rangle_{+(\beta)} & \text{for } a + b + c = 2h, \end{cases} \quad (4.1)$$

where  $v(a, b, d) = |b - d| + b - d - |a + b - d|$ .

The explicit form of on-shell three point vertex is

$$(\Gamma_{ab}^c)^2 = \Gamma^{(h)} < 2u_{ab}^c - 1 >_{+(0)} \prod_{\substack{x=|a-b|+1 \\ \text{step 2}}}^{u_{ab}^c-3} \frac{< u_{ab}^c + x >_{+(0)}}{< u_{ab}^c - x >_{+(0)}}, \quad (4.2)$$

where

$$\Gamma^{(h)} = \begin{cases} 2\pi^2 / \sin \frac{\pi}{h} & \text{for perturbed conformal} \\ -\left(\cos \frac{\pi}{h} - \cos \frac{\pi}{h}(B-1)\right) / \sin \frac{\pi}{h} & \text{for Affine Toda.} \end{cases}$$

For the  $A_N$  theories,  $u_{ab}^c = h - |a + b - h|$  for  $a + b + c = 0 \bmod h$ .

$$< x >_{+(0)} = \begin{cases} -\left(\cos \frac{\pi}{h}x - \cos \frac{\pi}{h}\right) / 2\pi^2 & \text{for perturbed conformal} \\ \left(\cos \frac{\pi}{h}x - \cos \frac{\pi}{h}\right) / \left(\cos \frac{\pi}{h}x - \cos \frac{\pi}{h}(B-1)\right) & \text{for Affine Toda.} \end{cases}$$

Especially for the Affine  $A_2$  theory,  $(\Gamma)^2 = \Gamma^{(3)} < 3 >_{+(0)}$  which agree with the previous result.

The on-shell three point vertex  $\Gamma_{ab}^c$  can be expressed as

$$(\Gamma_{ab}^c)^2 = \begin{cases} (\Gamma_c / \Gamma_a \Gamma_b)^2 & \text{for } a + b + c = h \\ (\Gamma_{\bar{c}} / \Gamma_{\bar{a}} \Gamma_{\bar{b}})^2 & \text{for } a + b + c = 2h, \end{cases} \quad (4.3)$$

where  $\Gamma_a = \prod_{d=1}^{a-1} \Gamma_{1d}^{h-d-1}$ .

The  $S$ -matrices for the  $A_N$  theories ( $N \geq 3$ ) contain double poles. The singularity of form factors are parameterized by the following functions

$$W_{ab}(\beta) = \prod_{\substack{x=|a-b|+2 \\ \text{step 2}}}^{u_{ab}^c-2\delta_{a+b,h}} (x)_+(-x)_+. \quad (4.4)$$

In the above factorization, we allow the case  $c = 0$  i.e.  $a+b = h$ . For  $a+b = h$ , the constant  $u_{ab}^c$  is taken that  $u_{ab}^0 = h$ . Corresponding to the double pole of the  $S$ -matrices, additional simple pole appears at (relative) rapidity  $\beta = i\frac{\pi}{h}(u_{ab}^c - 2k)$   $k = 1, \dots, \frac{1}{2}(u_{ab}^c - |a-b|) - 1$ . So we must determine these additional pole structure of the form factors.

From now on, we consider the Affine cases for definiteness. The one particle form factor  $F_a$  is constant.

$$F_a = < 0 | \phi_{\bar{a}}(0) | \beta >_a = \frac{1}{\sqrt{2}}.$$

The elementary two particle form factor is given as

$$F_{1a}(\beta) = -\Gamma_{1a}^{h-a-1} F_{a+1} \frac{\sin \theta_{1a}^{n-a-1}}{\cosh \beta - \cos \theta_{1a}^{h-a-1}} \frac{F_{1a}^{(min)}(\beta)}{F_{1a}^{(min)}(i\theta_{1a}^{h-a-1})} \text{ for } a \neq N.$$

$$F_{1N}(\beta) = F_{1N}^\Theta(\beta) = \frac{\pi}{2} M_1^2 \frac{F_{1N}^{(min)}(\beta)}{F_{1N}^{(min)}(i\pi)}.$$

These two-body form factors play the role of the initial conditions of the recursion equations.

In order to fix the additional simple pole structure, we analyze some low order form factors.

Solving the recursion relation for  $F_{11a}$ , we determined the explicit form of  $F_{2a}$  for  $a < N - 1$ ,

$$F_{2a}(\beta) = -\Gamma_{2a}^{h-a-2} F_{a+2} \frac{F_{2a}^{(min)}(\beta)}{F_{2a}^{(min)}(i\frac{\pi}{h}(a+2))} \times \left\{ \frac{\sin \frac{\pi}{h}(a+2)}{\cosh \beta - \cos \frac{\pi}{h}(a+2)} - \frac{\sin \frac{\pi}{h}(a+2)}{\cosh \beta - \cos \frac{\pi}{h}a} \left( \frac{\cos \frac{\pi}{h} - \cos \frac{\pi}{h}(B-1)}{\cos \frac{3\pi}{h} - \cos \frac{\pi}{h}(B-1)} \right) \right\}.$$

Also for  $a = N - 1 = \bar{2}$ ,

$$F_{22}(\beta) = \frac{\pi}{2} M_2^2 \frac{F_{22}^{(min)}(\beta)}{F_{22}^{(min)}(i\pi)} \left\{ 1 - 2 \sin^2 \frac{\pi}{h} \left( \frac{\cos \frac{\pi}{h} - \cos \frac{\pi}{h}(B-1)}{\cos \frac{3\pi}{h} - \cos \frac{\pi}{h}(B-1)} \right) \times \left( \frac{1}{\cosh \beta - \cos \frac{\pi}{h}(h-2)} + \frac{1}{1 + \cos \frac{\pi}{h}(h-2)} \right) \right\}.$$

One can show that

$$\frac{F_{ab}^{(min)}\left(i\frac{\pi}{h}(u_{ab}^c - 2k)\right)}{F_{ab}^{(min)}\left(i\frac{\pi}{h}u_{ab}^c\right)} = \frac{F_{l(u_{ab}^c-l)}^{(min)}\left(i\frac{\pi}{h}|a-b|\right)}{F_{l(u_{ab}^c-l)}^{(min)}\left(i\frac{\pi}{h}u_{ab}^c\right)} \quad k = 1, \dots, \frac{1}{2}(u_{ab}^c - |a-b|) - 1, \quad (4.5)$$

where  $l = k$  for  $a + b \leq h$  and  $l = \bar{k}$  for  $a + b > h$ .

Using eq.(4.5), we can see that

$$-i \operatorname{res}_{\beta=i\frac{\pi}{h}a} F_{2a}(\beta) = \Gamma_{11}^{h-2} \Gamma_{1a}^{h-a-1} F_{1(a+1)} \left( i\frac{\pi}{h}(a-2) \right). \quad (4.6)$$

For the case of  $a = N - 1$ , the factor  $\Gamma_{1(N-1)}^1 / \Gamma_{11}^{N-1}$  appears. From the form of  $\Gamma_{ab}^c$ , it holds that  $(\Gamma_{1(N-1)}^1)^2 = (\Gamma_{11}^{N-1})^2$ . In showing eq.(4.6), we take the phase  $\Gamma_{1(N-1)}^1 / \Gamma_{11}^{N-1} = 1$ .

In general, it is expected that

$$-i \operatorname{res}_{\beta=i\frac{\pi}{h}(u_{ab}^c-2k)} F_{ab}(\beta) = \Gamma_{lm_1}^{h-l-m_1} \Gamma_{m_2(m_1-l)}^{h-u_{ab}^c+l} F_{l(u_{ab}^c-l)} \left( i\frac{\pi}{h}|a-b| \right) \quad k = 1, \dots, m_1 - 1, \quad (4.7)$$

where  $l = k$  for  $a + b \leq h$ ,  $l = \bar{k}$  for  $a + b > h$ ,  $m_1 = \frac{1}{2}(u_{ab}^c - |a-b|)$  and  $m_2 = \frac{1}{2}(u_{ab}^c + |a-b|)$ . And  $\Gamma_{1a}^{h-a-1} = \Gamma_{1(h-a-1)}^a$ . We checked the above equation for the case of  $a = 3$ .

The additional simple pole structure is determined as follows:

$$\begin{aligned} & -i \operatorname{res}_{\beta'=\beta+i\frac{\pi}{h}(a+b-2k)} F_{abd_1 \dots d_n}(\beta', \beta, \beta_1, \dots, \beta_n) \\ & = \Gamma_{k(a-k)}^{h-a} \Gamma_{(a-k)b}^{h-a-b+k} F_{k(a+b-k)d_1 \dots d_n} \left( \beta + i\frac{\pi}{h}(b-k), \beta + i\frac{\pi}{h}(a-k), \beta_1, \dots, \beta_n \right), \end{aligned} \quad (4.8)$$

for  $a + b \leq h$  and  $a \leq b$ .

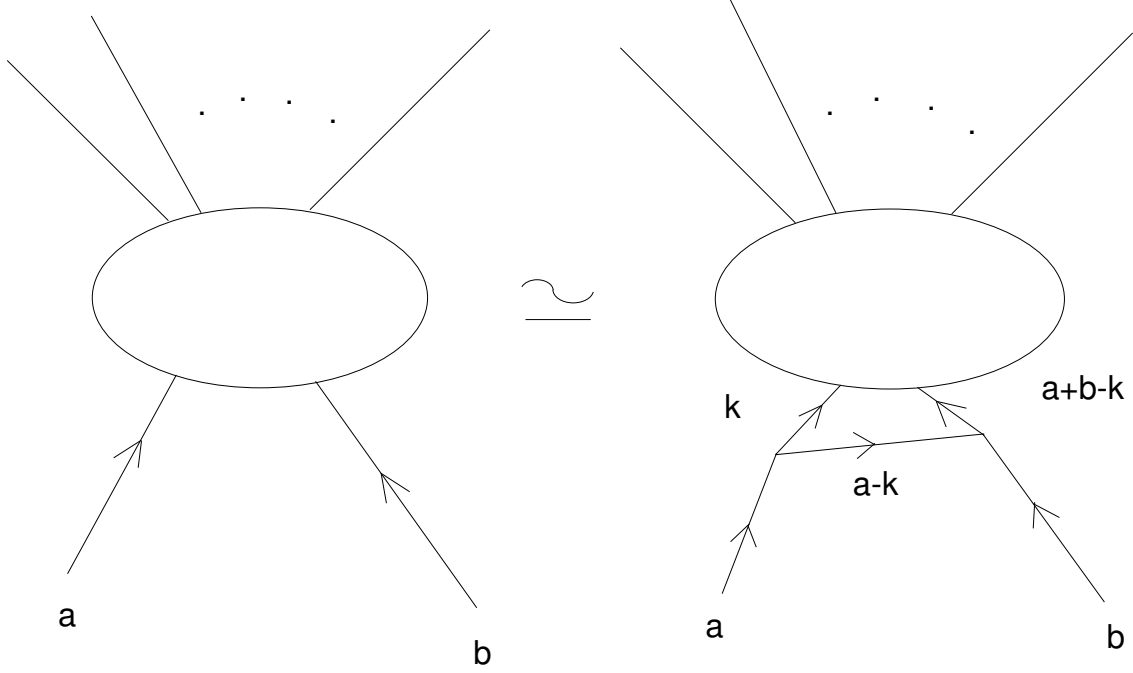


Figure : The additional pole structure for  $a + b \leq h$  and  $a \leq b$ .

Similar relation holds for other cases. Above pole structure is similar to that of  $A_{2N}^{(2)}$  theories [2].

Using eq.(4.7) recursively, the two-body form factors are determined as

$$F_{ab}(\beta) = \sin \theta_{ab}^c \Gamma_{ab}^c F_{\bar{c}} \frac{F_{ab}^{(min)}(\beta)}{F_{ab}^{(min)}(i\theta_{ab}^c)} \sum_{l_0=0}^{\frac{1}{2}(u_{ab}^c - |a-b|) - 1} \frac{B_{ab;l_0}}{\cosh \beta - \cos \frac{\pi}{h}(u_{ab}^c - 2l_0)}, \quad (4.9)$$

where

$$B_{ab;l_0} = \sum_{n \geq 0} \left(-\frac{1}{2}\right)^n \sum_{l_1=1}^{l_0-1} \sum_{l_2=1}^{l_1-1} \cdots \sum_{l_n=1}^{l_{n-1}-1} \frac{\prod_{j=0}^n \sin \frac{\pi}{h}(u_{ab}^c - 2l_j) \left(\Gamma_{l_j(l_{j-1}-l_j)}^{h-l_{j-1}}\right)^2}{\prod_{j=1}^n \sin \frac{\pi}{h}(l_{j-2} - l_j) \sin \frac{\pi}{h}(u_{ab}^c - l_{j-2} - l_j)}. \quad (4.10)$$

Here  $l_{-1} = \frac{1}{2}(u_{ab}^c - |a-b|)$  and  $B_{ab;0} = 1$ .

Also

$$\begin{aligned} F_{a\bar{a}}(\beta) &= \frac{\pi}{2} M_a^2 \frac{F_{a\bar{a}}^{(min)}(\beta)}{F_{a\bar{a}}^{(min)}(i\pi)} \\ &\times \left\{ 1 - \sum_{l_0=1}^{m_1-1} B_{a;l_0} \left( \frac{1}{\cosh \beta - \cos \frac{\pi}{h}(h - 2l_0)} + \frac{1}{1 + \cos \frac{\pi}{h}(h - 2l_0)} \right) \right\}, \quad (4.11) \end{aligned}$$



where

$$B_{a;l_0} = \left( \frac{\sin \frac{\pi}{h} l_0}{\sin \frac{\pi}{h} a} \Gamma_{l_0(l_{-1}-l_0)}^{h-l_{-1}} \right)^2 \sin \frac{2\pi}{h} l_0 \\ \times \left\{ \sum_{n \geq 0} \sum_{l_1=1}^{l_0-1} \sum_{l_2=1}^{l_1-1} \cdots \sum_{l_n=1}^{l_{n-1}-1} \left( -\frac{1}{2} \right)^n \left( \frac{\sin \frac{\pi}{h} a}{\sin \frac{\pi}{h} l_{n-1}} \right)^2 \prod_{j=1}^n \frac{\sin \frac{2\pi}{h} l_j \left( \Gamma_{l_j(l_{j-1}-l_j)}^{h-l_{j-1}} \right)^2}{\sin \frac{\pi}{h} (l_{j-2} + l_j) \sin \frac{\pi}{h} (l_{j-2} - l_j)} \right\}.$$

Here  $l_{-1} = \min(a, \bar{a})$ .

## 5 Conclusions and discussion

We have derived the minimal two-particle form factors for ADE scattering theories. Using monodromy properties of the building blocks of minimal form factors, we have determined the functional equations satisfied by the minimal form factors. The function  $\lambda_{ab;d}^c$  will play a key role in constructing the solutions of the form factor bootstrap equations.

We have determined the parameterization function  $W_{ab}$  for the perturbed conformal theories and for the Affine Toda Field theories.

For the  $A_2$  Affine Toda theory, form factors are derived up to four-body, and the identifications of the fundamental operators have been done.

For the  $A_N$  theories, the form factors are divided into  $N + 1$  sectors. To each sector, there corresponds the elementary operator or the stress-energy operator. This is a generalization of the known result for the sinh-Gordon theory (i.e.  $A_1$  Toda theory) [6, 8] to the  $A_N$  cases. For the  $A_N$  theories, we have determined the two-particle form factors. Also, the additional simple pole structure of form factors has been determined.

The determination of higher order form factors remains to be solved.

Before concluding this article, we state a relation between roots and equivalence relation of weights.

Dorey's fusion rule for the simply-laced theories [14] is that the fusion process  $a \times b \rightarrow \bar{c}$  occurs if

$$w^{\xi(a)} \mu^{(a)} + w^{\xi(b)} \mu^{(b)} + w^{\xi(c)} \mu^{(c)} = 0, \quad (5.1)$$

for some integer  $\xi(a)$ ,  $\xi(b)$  and  $\xi(c)$ . The particle  $b$  is the anti-particle of  $a$  if

$$w^{\xi(a)} \mu^{(a)} + w^{\xi(b)} \mu^{(b)} = 0. \quad (5.2)$$

for some integer  $\xi(a)$  and  $\xi(b)$ .

We define 'fusion base' vector  $e_{ab}^c = \mu^{(a)} + \mu^{(b)} - \mu^{(c)}$  or  $e_{ab}^0 = \mu^{(a)} + \mu^{(b)}$  if eq.(5.1) or eq.(5.2) is satisfied and  $e_{ab}^c = 0$  otherwise.

For the fusion  $a \times b \rightarrow \bar{c}$ , if its fusion vector is expanded in simple roots

$$e_{ab}^c = \mu^{(a)} + \mu^{(b)} - \mu^{(\bar{c})} \\ = \sum_{d=1}^r m_d \alpha_d, \quad (5.3)$$

then the coefficient  $m_d$  take integer values. This is equivalent to the statement that for  $a \times b \rightarrow \bar{c}$ , following condition is necessary:

$$(C^{-1})^{ad} + (C^{-1})^{bd} - (C^{-1})^{\bar{c}d} \in \mathbf{Z}.$$

Here  $C$  is the Cartan matrix of the algebra. For the  $E_8$  algebra, all elements of  $C^{-1}$  are integers, so the above condition is always satisfied for any  $(a, b, c)$ . Only for the  $A_N$  algebra, above condition is also sufficient.

Using the explicit form of the inverse of the Cartan matrix for the  $A_N$  theories,

$$(C^{-1})^{ab} = \frac{1}{N+1} \min(a, b)(N+1 - \max(a, b)),$$

one can show that the above condition is equivalent to  $a + b + c = 0 \bmod N + 1$ .

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